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The complex Monge-Ampère equation
on compact Hermitian manifolds[1]

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Abstract

We show that, up to scaling, the complex Monge-Ampère equation on compact Hermitian manifolds always admits a smooth solution.

1 Introduction

Let (M, g) be a compact Hermitian manifold of complex dimension $n \geq 2$ and write ω for the corresponding real $(1,1)$ form

$$\omega = \sqrt{-1} \sum_{i,j} g_{i\overline{j}} dz^i \wedge d\overline{z^j}.$$

For a smooth real-valued function F on M, consider the complex Monge-Ampère equation

$$(\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^F \omega^n, \quad \text{with}$$
$$\omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0, \quad \sup_M \varphi = 0, \tag{1.1}$$

for a real-valued function φ.

Our main result is as follows.

Main Theorem *Let φ be a smooth solution of the complex Monge-Ampère equation (1.1). Then there are uniform C^∞ a priori estimates on φ depending only on (M, ω) and F.*

A corollary of this is that we can solve (1.1) uniquely after adding a constant to F, or equivalently, up to scaling the volume form $e^F \omega^n$.

Corollary 1. *For every smooth real-valued function F on M there exists a unique real number b and a unique smooth real-valued function φ on M solving*

$$(\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^{F+b} \omega^n, \quad \text{with}$$
$$\omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0, \quad \sup_M \varphi = 0. \tag{1.2}$$

In the case of ω Kähler, that is when $d\omega = 0$, this result is precisely the celebrated Calabi Conjecture [Ca] proved by Yau [Ya]. We note here that if ω satisfies

$$\partial\overline{\partial}\omega^k = 0, \quad \text{for } k = 1, 2, \tag{1.3}$$

(in particular if ω is closed) then the constant b must equal

$$\log \frac{\int_M \omega^n}{\int_M e^F \omega^n}.$$

We mention now some special cases where the results of the Main Theorem and Corollary are already known. Cherrier [Ch] gave a proof when the complex dimension is two or if ω is balanced, that is, $d(\omega^{n-1}) = 0$ (an alternative proof was very recently given in [TW]). In addition, Cherrier [Ch] dealt with the case of conformally Kähler and considered a technical assumption which is slightly weaker than balanced, see also the related work of Hanani [Ha]. Guan-Li [GL] gave a proof under the assumption (1.3). For further background we refer the reader to [TW] and the references therein.

As the reader will see in the proof below, we note that the key L^∞ bound of φ in the Main Theorem follows from combining a lemma of [Ch] with some recent estimates of the authors [TW].

Finally, we remark that one can give a geometric interpretation of (1.2) in terms of the first Chern class $c_1(M)$ of M. We denote by $\mathrm{Ric}(\omega)$ the first Chern form of the Chern connection of ω, which is a closed form cohomologous to $c_1(M)$. We then consider the real Bott-Chern space $H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$ of closed real $(1,1)$ forms modulo the image of $\sqrt{-1}\partial\overline{\partial}$ acting on real functions. It has a natural surjection to the familiar space $H^{1,1}(M, \mathbb{R})$, which is an isomorphism if and only if $b_1(M) = 2h^{0,1}$ [G2] (in particular if M is Kähler). The form $\mathrm{Ric}(\omega)$ determines a class $c^{\mathrm{BC}}_1(M)$ in $H^{1,1}_{\mathrm{BC}}(M, \mathbb{R})$ which maps to the usual first Chern class $c_1(M)$ via the above surjection. Then from our main theorem we get the following Hermitian version of the Calabi conjecture (see also a related question of Gauduchon [G2, IV.5]):

Corollary 2. *Every representative of the first Bott-Chern class $c^{\mathrm{BC}}_1(M)$ can be represented as the first Chern form of a Hermitian metric of the form $\omega + \sqrt{-1}\partial\overline{\partial}\varphi$.*

To see why this holds, just notice that (1.2) holds for some constant b if and only if

$$\mathrm{Ric}(\omega + \sqrt{-1}\partial\overline{\partial}\varphi) = \mathrm{Ric}(\omega) - \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}F, \tag{1.4}$$

and that by definition every form representing $c^{\mathrm{BC}}_1(M)$ can be written as $\mathrm{Ric}(\omega) - \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}F$ for some function F. We note here that in the case $n = 2$ [TW, Corollary 2] gives a criterion to decide which representatives of $c_1(M)$ can be written in this form.

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2 Proof of the Main Theorem

By the results of [Ch], [GL], [Zh] it suffices to obtain a uniform bound of φ in the L^∞ norm. Indeed, by extending the second order estimate on φ of Yau [Ya] (and Aubin [Au]), Cherrier [Ch] has shown, for general ω, that a uniform L^∞ bound on φ implies that the metric $\omega + \sqrt{-1}\partial\overline{\partial}\varphi$ is uniformly equivalent to ω. Moreover, generalizing Yau's third order estimate [Ya], Cherrier shows that given this one can then bound $\omega + \sqrt{-1}\partial\overline{\partial}\varphi$ in C^1. Higher order estimates then follow from standard elliptic theory. A similar second order estimate was also proved by Guan-Li [GL] and Zhang [Zh] for general ω, and sharpened in [TW] in the cases of $n = 2$ or ω balanced. It is also possible to avoid the third order estimate by using the Evans-Krylov theory, as in [GL] and [TW].

We remark that our L^∞ bound on φ depends only on (M, ω) and $\sup_M F$, as in Yau's estimate for the Kähler case [Ya]. In particular, the L^∞ bound does not depend on $\inf_M F$. In the course of the proof, we say that a constant is *uniform* if it depends only on the data (M, ω) and $\sup_M F$. We will often write such a constant as C, which may differ from line to line. If we say that a constant depends only on a quantity Q then we mean that it depends only on Q, (M, ω) and $\sup_M F$.

Our goal is thus to give a uniform bound for φ. We begin with a lemma which can be found in [Ch]. For the convenience of the reader, we provide a proof. We use the notation of exterior products instead of the multilinear algebra calculations of [Ch].

Lemma 2.1 *There are uniform constants* C, p_0 *such that for all* $p \geq p_0$ *we have*

$$\int_M |\partial e^{-\frac{p}{2}\varphi}|_g^2 \omega^n \leq Cp \int_M e^{-p\varphi}\omega^n.$$

Proof. From now on we will use the shorthand $\omega_\varphi = \omega + \sqrt{-1}\partial\overline{\partial}\varphi$. Let α be the $(n-1, n-1)$-form given by

$$\alpha = \sum_{k=0}^{n-1} \omega_\varphi^k \wedge \omega^{n-k-1}.$$

We compute, using the equation (1.1) and integrating by parts,

$$
\begin{aligned}
C\int_M e^{-p\varphi}\omega^n &\geq \int_M e^{-p\varphi}(\omega_\varphi^n - \omega^n) \\
&= \int_M e^{-p\varphi}\sqrt{-1}\partial\overline{\partial}\varphi \wedge \alpha \\
&= p\int_M e^{-p\varphi}\sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \alpha + \int_M e^{-p\varphi}\sqrt{-1}\,\overline{\partial}\varphi \wedge \partial\alpha. \qquad (2.1)
\end{aligned}
$$

The first term on the right hand side of (2.1) is positive, and we are going to use part of it to deal with the second one. Notice that

$$\partial\alpha = n\sum_{k=0}^{n-2} \omega_\varphi^k \wedge \omega^{n-k-2} \wedge \partial\omega.$$

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Since $\partial\omega$ is a fixed tensor, there is a constant C so that for any $\varepsilon > 0$ and any k we have the following elementary pointwise inequality

$$\left|\frac{\sqrt{-1}\,\overline{\partial}\varphi \wedge \partial\omega \wedge \omega_\varphi^k \wedge \omega^{n-k-2}}{\omega^n}\right|$$

$$\leq \frac{C}{\varepsilon}\frac{\sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^k \wedge \omega^{n-k-1}}{\omega^n} + \varepsilon C \frac{\omega_\varphi^k \wedge \omega^{n-k}}{\omega^n}, \qquad (2.2)$$

that the reader can verify by choosing local coordinates at a point that make ω the identity and ω_φ diagonal. Applying (2.2) we have for any $\varepsilon > 0$ and any p,

$$-\int_M e^{-p\varphi}\sqrt{-1}\,\overline{\partial}\varphi \wedge \partial\alpha = -n\sum_{k=0}^{n-2}\int_M e^{-p\varphi}\sqrt{-1}\,\overline{\partial}\varphi \wedge \omega_\varphi^k \wedge \omega^{n-k-2} \wedge \partial\omega$$

$$\leq \frac{C}{\varepsilon}\sum_{k=0}^{n-2}\int_M e^{-p\varphi}\sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^k \wedge \omega^{n-k-1}$$

$$+ \varepsilon C\sum_{k=0}^{n-2}\int_M e^{-p\varphi}\omega_\varphi^k \wedge \omega^{n-k}.$$

Now if we choose $p_0/2 \geq C/\varepsilon$ we see that if $0 < \varepsilon \leq 1$ then for $p \geq p_0$,

$$-\int_M e^{-p\varphi}\sqrt{-1}\,\overline{\partial}\varphi \wedge \partial\alpha \leq \frac{p}{2}\int_M e^{-p\varphi}\sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \alpha + C\int_M e^{-p\varphi}\omega^n$$

$$+ \varepsilon C\sum_{k=1}^{n-2}\int_M e^{-p\varphi}\omega_\varphi^k \wedge \omega^{n-k}.$$

Combining this with (2.1) we see that for any $0 < \varepsilon < 1$ there exists p_0 depending only on ε such that for $p \geq p_0$,

$$\frac{p}{2}\int_M e^{-p\varphi}\sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \alpha \leq C\int_M e^{-p\varphi}\omega^n + \varepsilon C\sum_{k=1}^{n-2}\int_M e^{-p\varphi}\omega_\varphi^k \wedge \omega^{n-k}. \qquad (2.3)$$

We now claim the following. There exist uniform constants $C_2,\ldots,C_n$ and ε_0 such that for all ε with $0 < \varepsilon \leq \varepsilon_0$, there exists a constant p_0 depending only on ε such that for all $p \geq p_0$ we have for $i = 2,\ldots,n$,

$$\frac{p}{2^{i-1}}\int_M e^{-p\varphi}\sqrt{-1}\partial\varphi \wedge \overline{\partial}\varphi \wedge \alpha \leq C_i\int_M e^{-p\varphi}\omega^n + \varepsilon C_i\sum_{k=1}^{n-i}\int_M e^{-p\varphi}\omega_\varphi^k \wedge \omega^{n-k}. \qquad (2.4)$$

Given the claim, the lemma follows. Indeed once we have the statement with $i = n$

then, fixing $\varepsilon = \varepsilon_0$ we have for $p \geq p_0$,

$$
\begin{aligned}
\int_M |\partial e^{-\frac{p}{2}\varphi}|_g^2 \omega^n &= \frac{np^2}{4} \int_M e^{-p\varphi} \sqrt{-1} \partial\varphi \wedge \overline{\partial}\varphi \wedge \omega^{n-1} \\
&\leq \frac{np^2}{4} \int_M e^{-p\varphi} \sqrt{-1} \partial\varphi \wedge \overline{\partial}\varphi \wedge \alpha \\
&\leq n2^{n-3} C_n p \int_M e^{-p\varphi} \omega^n,
\end{aligned}
$$

as required.

We will prove the claim by induction on i. By (2.3) we have already proved the statement for $i = 2$. So we assume the induction statement (2.4) for i, and prove it for $i + 1$. We compute

$$
\begin{aligned}
\varepsilon C_i \sum_{k=1}^{n-i} \int_M e^{-p\varphi} \omega_\varphi^k \wedge \omega^{n-k} &= \varepsilon C_i \sum_{k=1}^{n-i} \int_M e^{-p\varphi} \omega_\varphi^{k-1} \wedge \omega^{n-k+1} \\
&\quad + \varepsilon C_i \sum_{k=1}^{n-i} \int_M e^{-p\varphi} \sqrt{-1} \partial\overline{\partial}\varphi \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k} \\
&= A_1 + A_2,
\end{aligned}
\tag{2.5}
$$

where

$$
A_1 = \varepsilon C_i \sum_{k=0}^{n-i-1} \int_M e^{-p\varphi} \omega_\varphi^k \wedge \omega^{n-k}, \quad A_2 = \varepsilon C_i \sum_{k=0}^{n-i-1} \int_M e^{-p\varphi} \sqrt{-1} \partial\overline{\partial}\varphi \wedge \omega_\varphi^k \wedge \omega^{n-k-1}.
$$

The term A_1 is already acceptable for the induction. For A_2 we integrate by parts to obtain

$$
\begin{aligned}
A_2 &= \varepsilon C_i p \sum_{k=0}^{n-i-1} \int_M e^{-p\varphi} \sqrt{-1} \partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^k \wedge \omega^{n-k-1} \\
&\quad + \varepsilon C_i \sum_{k=1}^{n-i-1} k \int_M e^{-p\varphi} \sqrt{-1} \overline{\partial}\varphi \wedge \omega_\varphi^{k-1} \wedge \omega^{n-k-1} \wedge \partial\omega \\
&\quad + \varepsilon C_i \sum_{k=0}^{n-i-1} (n-k-1) \int_M e^{-p\varphi} \sqrt{-1} \overline{\partial}\varphi \wedge \omega_\varphi^k \wedge \omega^{n-k-2} \wedge \partial\omega \\
&= B_1 + B_2 + B_3,
\end{aligned}
\tag{2.6}
$$

where

$$B_1 = \varepsilon C_i p \sum_{k=0}^{n-i-1} \int_M e^{-p\varphi} \sqrt{-1} \partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^k \wedge \omega^{n-k-1}$$

$$B_2 = \varepsilon C_i \sum_{k=0}^{n-i-2} (k+1) \int_M e^{-p\varphi} \sqrt{-1}\, \overline{\partial}\varphi \wedge \omega_\varphi^k \wedge \omega^{n-k-2} \wedge \partial\omega$$

$$B_3 = \varepsilon C_i \sum_{k=0}^{n-i-1} (n-k-1) \int_M e^{-p\varphi} \sqrt{-1}\, \overline{\partial}\varphi \wedge \omega_\varphi^k \wedge \omega^{n-k-2} \wedge \partial\omega.$$

Choosing ε_0 such that $\varepsilon_0 C_i < 2^{-i-1}$ we have for $\varepsilon < \varepsilon_0$ and $p \geq p_0$,

$$B_1 \leq \frac{p}{2^{i+1}} \int_M e^{-p\varphi} \sqrt{-1} \partial\varphi \wedge \overline{\partial}\varphi \wedge \alpha. \tag{2.7}$$

For the terms B_2 and B_3 we use again (2.2) to obtain

$$B_2 + B_3 \leq n C_i C \sum_{k=0}^{n-i-1} \int_M e^{-p\varphi} \sqrt{-1} \partial\varphi \wedge \overline{\partial}\varphi \wedge \omega_\varphi^k \wedge \omega^{n-k-1}$$
$$+ \varepsilon^2 n C_i C \sum_{k=0}^{n-i-1} \int_M e^{-p\varphi} \omega_\varphi^k \wedge \omega^{n-k}. \tag{2.8}$$

Notice that the second term on the right hand side of (2.8) is acceptable for the induction. Moreover, we may assume that $p_0 \geq 2^{i+1} n C_i C$ and thus for $p \geq p_0$,

$$B_2 + B_3 \leq \frac{p}{2^{i+1}} \int_M e^{-p\varphi} \sqrt{-1} \partial\varphi \wedge \overline{\partial}\varphi \wedge \alpha$$
$$+ \varepsilon^2 n C_i C \sum_{k=0}^{n-i-1} \int_M e^{-p\varphi} \omega_\varphi^k \wedge \omega^{n-k}. \tag{2.9}$$

Combining the inductive hypothesis (2.4) with (2.5), (2.6), (2.7), (2.9) we obtain for $p \geq p_0$,

$$\frac{p}{2^i} \int_M e^{-p\varphi} \sqrt{-1} \partial\varphi \wedge \overline{\partial}\varphi \wedge \alpha \leq C_{i+1} \int_M e^{-p\varphi} \omega^n + \varepsilon C_{i+1} \sum_{k=1}^{n-i-1} \int_M e^{-p\varphi} \omega_\varphi^k \wedge \omega^{n-k}, \tag{2.10}$$

completing the inductive step. This finishes the proof of the claim and thus the lemma. $\square$

We now complete the proof of the Main Theorem. Using Lemma 2.1 and the Sobolev inequality, we have for $\beta = \frac{n}{n-1} > 1$,

$$\left(\int_M e^{-p\beta\varphi} \omega^n \right)^{1/\beta} \leq C \left(\int_M |\partial e^{-\frac{p}{2}\varphi}|^2 \omega^n + \int_M e^{-p\varphi} \omega^n \right)$$
$$\leq Cp \int_M e^{-p\varphi} \omega^n,$$

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for all $p \geq p_0$. Thus
$$\|e^{-\varphi}\|_{L^{p\beta}} \leq C^{1/p} p^{1/p} \|e^{-\varphi}\|_{L^p}.$$

Since this holds for all $p \geq p_0$, we can iterate this estimate in a standard way to obtain

$$\|e^{-\varphi}\|_{L^{\infty}} \leq C \|e^{-\varphi}\|_{L^{p_0}},$$

which is equivalent to

$$e^{-p_0 \inf_M \varphi} \leq C \int_M e^{-p_0 \varphi} \omega^n.$$

We now make use of a result from [TW]:

Lemma 2.2 *Let f be a smooth function on (M, ω). Write $d\mu = \omega^n / \int_M \omega^n$. If there exists a constant C_1 such that*

$$e^{-\inf_M f} \leq e^{C_1} \int_M e^{-f} d\mu, \tag{2.11}$$

then

$$\left|\{f \leq \inf_M f + C_1 + 1\}\right| \geq \frac{e^{-C_1}}{4}, \tag{2.12}$$

where $|\cdot|$ denotes the volume of the set with respect to $d\mu$.

Proof. See [TW, Lemma 3.2]. $\qquad\square$

Applying this lemma to $f = p_0 \varphi$ we see that there exist uniform constants $C, \delta > 0$ so that

$$\left|\{\varphi \leq \inf_M \varphi + C\}\right| \geq \delta. \tag{2.13}$$

We remark that, in [TW], the bound (2.13) is established whenever one has the improved second order estimate,

$$\mathrm{tr}_\omega \omega_\varphi \leq C e^{A(\varphi - \inf_M \varphi)}, \tag{2.14}$$

for uniform A and C. It is shown in [TW] that (2.14) holds if $n = 2$ or ω is balanced.

The L^∞ bound on φ, and hence the Main Theorem, now follow from the arguments of [TW]. However, we include an outline of these arguments for the reader's convenience. Recall that, from [G1], if (M, ω) is a compact Hermitian manifold then there exists a unique smooth function $u : M \to \mathbb{R}$ with $\sup_M u = 0$ such that the metric $\omega_{\mathrm{G}} = e^u \omega$ is *Gauduchon*, that is, satisfies

$$\partial \overline{\partial}(\omega_{\mathrm{G}}^{n-1}) = 0. \tag{2.15}$$

Writing Δ_{G} for the complex Laplacian associated to ω_{G} (which differs from the Levi-Civita Laplacian in general), we have the following lemma.

Lemma 2.3 *Let M be a compact complex manifold of complex dimension n with a Gauduchon metric $\omega_{\rm G}$. If ψ is a smooth nonnegative function on M with*

$$\Delta_{\rm G}\psi \geq -C_0$$

then there exist constants C_1 and C_2 depending only on $(M, \omega_{\rm G})$ and C_0 such that:

$$\int_M |\partial\psi^{\frac{p+1}{2}}|^2_{\omega_{\rm G}}\omega_{\rm G}^n \leq C_1 p \int_M \psi^p \omega_{\rm G}^n \quad \text{for all } p \geq 1, \tag{2.16}$$

and

$$\sup_M \psi \leq C_2 \max\left\{\int_M \psi\,\omega_{\rm G}^n, 1\right\}. \tag{2.17}$$

Proof. Although [TW, Lemma 3.4] is stated for complex dimension 2, the same proof works for any dimension. $\qquad\square$

We apply Lemma 2.3 to the function $\psi = \varphi - \inf_M \varphi$, which satisfies $\Delta_{\rm G}\psi = e^{-u}\Delta\psi > -C$, where Δ is the complex Laplacian with respect to ω. In light of (2.17), once we bound the L^1 norm of ψ the Main Theorem follows. Denoting by $\underline{\psi}$ the average of ψ with respect to $\omega_{\rm G}^n$ we obtain from the Poincaré inequality and (2.16) with $p = 1$,

$$\|\psi - \underline{\psi}\|_{L^2} \leq C\left(\int_M |\partial\psi|^2_{\omega_{\rm G}}\omega_{\rm G}^n\right)^{1/2} \leq C\|\psi\|_{L^1}^{1/2}. \tag{2.18}$$

In (2.18) and the following we are using L^q norms with respect to the volume form $\omega_{\rm G}^n$, which are equivalent to L^q norms with respect to $d\mu$. Using (2.13) we see that the set $S := \{\psi \leq C\}$ satisfies $|S|_{\rm G} \geq \delta$ for a uniform $\delta > 0$, where $|\cdot|_{\rm G}$ denotes the volume of a set with respect to $\omega_{\rm G}^n$. Hence

$$\frac{\delta}{\int_M \omega_{\rm G}^n}\int_M \psi\omega_{\rm G}^n = \delta\underline{\psi} \leq \int_S \underline{\psi}\omega_{\rm G}^n \leq \int_S (|\psi - \underline{\psi}| + C)\omega_{\rm G}^n \leq \int_M |\psi - \underline{\psi}|\omega_{\rm G}^n + C.$$

Then,

$$\|\psi\|_{L^1} \leq C(\|\psi - \underline{\psi}\|_{L^1} + 1) \leq C(\|\psi - \underline{\psi}\|_{L^2} + 1) \leq C(\|\psi\|_{L^1}^{1/2} + 1),$$

which shows that ψ is uniformly bounded in L^1. This completes the proof of the Main Theorem.

Finally we mention that corollary 1 follows from the argument of Cherrier [Ch], which uses results from [De]. Or for another proof, see [TW, Corollary 1].

References

[Au] Aubin, T. *Équations du type Monge-Ampère sur les variétés kählériennes compactes*, Bull. Sci. Math. (2) **102** (1978), no. 1, 63–95.

[Ca] Calabi, E. *On Kähler manifolds with vanishing canonical class*, in *Algebraic geometry and topology. A symposium in honor of S. Lefschetz*, pp. 78–89. Princeton University Press, Princeton, N. J., 1957.

[Ch] Cherrier, P. *Équations de Monge-Ampère sur les variétés Hermitiennes compactes*, Bull. Sc. Math (2) **111** (1987), 343–385.

[De] Delanoë, P. *Équations du type de Monge-Ampère sur les variétés Riemanniennes compactes, II*, J. Functional Anal. **41** (1981), 341–353.

[G1] Gauduchon, P. *Le théorème de l'excentricité nulle*, C. R. Acad. Sci. Paris **285** (1977), 387–390.

[G2] Gauduchon, P. *La 1-forme de torsion d'une variètè hermitienne compacte*, Math. Ann. **267** (1984), no. 4, 495–518.

[GL] Guan, B., Li, Q. *Complex Monge-Ampère equations on Hermitian manifolds*, preprint, arXiv:0906.3548.

[Ha] Hanani, A. *Équations du type de Monge-Ampère sur les variétés hermitiennes compactes*, J. Funct. Anal. **137** (1996), no. 1, 49–75.

[TW] Tosatti, V., Weinkove, B. *Estimates for the complex Monge-Ampère equation on Hermitian and balanced manifolds*, preprint, arXiv:0909.4496.

[Ya] Yau, S.-T. *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I*, Comm. Pure Appl. Math. **31** (1978), no.3, 339–411.

[Zh] Zhang, X. *A priori estimates for complex Monge-Ampère equation on Hermitian manifolds*, preprint 2009.

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